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Note

On the blocking sets in $S(3,6,22)$ and $S(4,7,23)$

Xiaomin Bao

Department of Mathematics, University of Manitoba, Winnipeg, Man., Canada R3T 2N2

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Abstract

The aim of this short note is to prove the uniqueness of certain blocking sets studied in Berardi (Ann. Discrete Math. 37 (1988) 31–42; J. Inf. Opti. Sci. 2 (1988) 263–298). Precisely, starting from a characterization contained in Berardi (1988) we prove that, up to isomorphism, in $S(3,6,22)$ there is exactly one blocking set having size nine, and in $S(4,7,23)$ there is exactly one minimal blocking set having six points on a block.

MSC: 05B05; 51E21

Keywords: Steiner system; Blocking set

1. Introduction

First of all, we introduce some terminologies and notations.

Definition 1.1. A *Steiner system* $S(t,k,v)$ is a pair $(\mathscr{S},\mathscr{B})$, where $\mathscr{S}$ is a v-set of elements called points, $\mathscr{B}$ is a family of k-sets called blocks, such that any fixed t-set is contained in exactly one element of $\mathscr{B}$.

Definition 1.2. A set of points of a Steiner system is called a blocking set if it contains no block, but intersects every block.

Definition 1.3. Let E be a set of points of $S(t,k,v)$, let t_i be the number of blocks that are i-secant to E, $i=0,1,\ldots,k$. If $\{i_1,\ldots,i_n\}$ is the set of all those i_j's $(0\leqslant i_j\leqslant k)$ such that $t_{i_j}\neq 0$ and $0\leqslant i_1<i_2<\cdots<i_n$, then we say that E is of type $(i_1,i_2,\ldots,i_n)$.

We recall the following:

E-mail address: bao@cc.umanitoba.ca (X. Bao).

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Result 1.1 (Berardi [1, Result 2.1]). Let B and B' be two blocks in $S(3,6,22)$. Then either $|B \cap B'| = 0$ or $|B \cap B'| = 2$.

Result 1.2 (Berardi [1, Lemmas 2.2 and 2.7]). Let B and B' be two blocks in $S(3,6,22)$. If $|B \cap B'| = 2$, then the type of $B \cup B'$ is $(0,2,3,4,6)$ with

$$t_0 = 4, \quad t_2 = 27, \quad t_3 = 32, \quad t_4 = 12, \quad t_6 = 2.$$

If $|B \cap B'| = 0$, then the type of $B \cup B'$ is $(2,4,6)$ with

$$t_2 = 30, \quad t_4 = 45, \quad t_6 = 2.$$

Result 1.3 (Berardi [1, Lemma 5.1]). Let B, B' be two blocks of $S(3,6,22)$ with $|B \cap B'| = 2$. Denote by a, b two points of $B' - B$. Let $\mathscr{E} = \{E_1, E_2, E_3, E_4\}$ be the set of the four blocks external to $B \cup B'$:

1. There exist two blocks S_1, S_2 which are 2-secant to $B \cup B'$ at a and b.
2. For every $x \in S_1$ (or S_2), with $x \neq a, b$ the two blocks E_i, E_j of $\mathscr{E}$ through x have the other common point y on S_1 (or S_2).
3. One of the two points outside $B \cup B' \cup E_1 \cup E_2$ is in S_1 and the other in S_2.

Result 1.4 (Berardi [2, Result 2.1]). Every block in $S(4,7,23)$ is a 7-set of type $(1,3,7)$.

Result 1.5 (Berardi [2, Lemma 2.5]). Let B, B' be two blocks of $S(4,7,23)$ with $|B \cap B'| = 3$. Fix $x \in B - B'$ and $y \in B' - B$. There are exactly three blocks D_1, D_2 and D_3 intersecting $B \cup B'$ only at x and y. Moreover $D_1 \cap D_2 \cap D_3 = \{x, y\}$.

Berardi [1,2] defined the following sets:

1. Use the same notations of Result 1.3. Fix $u \in B - B'$ and let z be the only point of S_2 outside $B \cup B' \cup E_1 \cup E_2$. Define

   $$N_1 := (B \cup B' - \{a, b, u\}) \cup \{x, z\}.$$

2. Let B, B' and B'' be three blocks in $S(3,6,22)$ with $|B \cap B' \cap B''| = 1$. Fix a point z of $B'' - (B \cup B')$, let y be one of the joint points of the two external blocks to $B \cup B' \cup \{z\}$. Define

   $$N_0 := [(B \cup B') - B''] \cup \{y, z\}.$$

3. $E_0 := \mathscr{S} - (B \triangle B')$, where B and B' are two blocks of $S(4,7,23)$ with $|B \cap B'| = 1$.
4. $E := (B - \{x\}) \cup (B' - \{y\} \cup \{a, b\})$, where B and B' are two blocks of $S(4,7,23)$ with $|B \cap B'| = 3$, $x \in B - B'$, $y \in B' - B$ and

   $$\{a, b\} \subset \{x_1, x_2, x_3 \mid \{x_k\} = D_i \cap D_j - (B \cup B'), \{i, j, k\} = \{1, 2, 3\}\},$$

   here D_1, D_2 and D_3 are the three blocks intersecting $B \cup B'$ only at x and y.

5. $E_1 = B \cup B' - \{o, w\}$, where B and B' are two blocks of $S(4,7,23)$ with $B \cap B' = \{o\}$ and $w \in B' - B$.

Berardi [1] proved that N_1 and N_0 are blocking sets of size nine in $S(3,6,22)$. In [2] Berardi proved that E_0, E and E_1 are blocking sets of size eleven in $S(4,7,23)$. Here we prove that N_1 and N_0 are same type of blocking set in $S(3,6,22)$; while E and E_1 are same type of blocking set in $S(4,7,23)$.

2. The proofs of our results

Let u_i denote the number of blocks of a $S(t,k,v)$ through a point on a blocking c-set C that intersect C in exactly i points. Suppose that $i \leqslant n < k$. Then u_i satisfy the following equalities:

$$\sum_{i=1}^{n} u_i = r_1, \qquad \sum_{i=2}^{n} u_i = c - 1. \tag{1}$$

Lemma 2.1. *Each blocking set C of size nine in $S(3,6,22)$ has at least a 5-secant block.*

Proof. On the contrary, suppose that in $S(3,6,22)$ there is a blocking set C of size nine, which has no 5-secant block. Then (2.2) of [1] implies that there are 17 blocks 1-secant to C. Moreover, from (1) it follows that C has at least 13 blocks tangent at each its point, a contradiction. □

Corollary 2.1. *The set N_0 is the unique blocking set of size nine in $S(3,6,22)$.*

Now, we prove:

Lemma 2.2. *A blocking set C of $S(4,7,23)$ with $|C| = 11$ cannot have only one 6-secant block.*

Proof. Assume, on the contrary, that a blocking set C with $|C| = 11$, has $t_6 = 1$. From (2.2) of [2] it follows that $t_1 = 21$. Moreover, from (1) it follows that C has at least 67 blocks tangent at each its point, a contradiction. □

Corollary 2.2. *The blocking sets E and E_1 are the same type of blocking set in $S(4,7,23)$.*

By Remark 2.8 in [2] and Corollary 2.2 we know that E_0 and E_1 are the only two non-equivalent blocking sets of size eleven in $S(4,7,23)$. So there are four non-equivalent blocking sets in $S(4,7,23)$.

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References

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[2] L. Berardi, Blocking sets in the large Mathieu designs, II: the case $S(4,7,23)$, J. Inf. Opti. Sci. 2 (1988) 263–278.